Intermittency effects in Burgers equation driven by thermal noise.

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Abstract

For the Burgers equation driven by thermal noise leading asymptotics of pair and high-order correlators of the velocity field are found for finite times and large distances. It is shown that the intermittency takes place: some correlators are much larger than their reducible parts. (Pis'ma v ZhETF, v.71, n.1, January 2000, in press, English Translation in JETP Letters, v.71, January 2000).

Intermittency implies strong non-Gaussianity of statistics of fluctuating fields. This phenomenon is shown by hydrodynamical systems in a state of developed turbulence[1, 2, 3]. In such far from equilibrium situations intermittency appears as prevalence of irreducible parts of some fourth order simultaneous correlators over reducible ones.

As for thermal equilibrium, irreducible parts of simultaneous correlators of local fluctuating fields turn out to be of the same order as their Gaussian parts even in critical region. This property is inherent in the renormalization group method which takes care of interaction of fluctuations through renormalization of local field and effective Hamiltonian [4].

In the recent paper [5] V.V.Lebedev disclosed that the picture can change drastically when we pass to time-dependent correlations of thermally fluctuating quantities. He found that in the low-temperature phase of two-dimensional systems of the Berezinskii-Kosterlitz-Thouless kind different-time correlation functions of the vortex charge density may greately exeed their own Gaussian part. In the same paper [5] the physical cause of such

the behaviour is pointed out: at low temperatures the non-simultaneous correlation function of every order in vicinity of a given poin are defined by a single rare fluctuation. One can conclude from this interpretation that the intermittency effects may emerge in equilibrium dynamics of a wide range of systems.

In the present paper I consider one-dimensional velocity field evolving according to the Burgers equation with the thermal noise term:

$$u_t + uu_x - \nu u_{xx} = \xi(t,x). \tag{1}$$

Here ν is the dissipation constant and $\xi(t,x)$ is random noise with Gaussian statistics and the pair correlator:

$$\langle \xi(t,x)\xi(t_1,x_1)\rangle = -\nu\beta^{-1}\delta''(x-x_1)\delta(t-t_1). \tag{2}$$

We will consider ν as being vanishingly small. The parameter β plays the role of inverse temperature, so the simultaneous stationary distribution function $\mathcal{P}[u]$ has the Gibbs form:

$$\mathcal{P}[u] = \mathcal{N}\exp\{-\beta\mathcal{F}[u]\}, \quad \mathcal{F}[u] = \int dx\, u^2(x). \tag{3}$$

Here $\mathcal{N}$ is the normalization constant. The expression:

$$\langle u(t,x)u(t,x')\rangle = (2\beta)^{-1}\delta(x-x'). \tag{4}$$

following from (3) correspond to a total absence of velocity correlation in spatially distant points in a given time moment. In the present paper some asympotics of various non-simultaneous correlators of the field $u(t,x)$ are found. The results obtained here show presence of intermittency effects in the equilibrium dynamics of the system (1).

Dynamical scaling exponent $z = 3/2$ for the problem (1)-(2) was discovered in the paper [6] using dimensional analysis and utilizing the Galilean invariance. In the paper [7] the absence of logarithmic divergencies for the spectrum $\omega \propto k^{3/2}$ was checked in every order of renormalized perturbation theory. Thus the function $F_2(T,x) =< u(T,x)u(0,0) >$ has $\beta x^3/T^2$ as a dimensionless argument. First we find the main (exponential) part of the asymptotics of the function F_2 at $\beta x^3/T^2 \gg 1$ and $\nu \to 0$. The latter limit means that the diffusion cannot set up correlation of velocity in the points

0 and x in a time T. The role of the noise in dynamics on the time interval $(0,T)$ is also negligible. Thus we can consider $u(0,y)$ as a functional of $u(T,x)$ and vice versa. The velocity statistics at the time moment T is Gaussian what allows us to represent the non-simultaleous correlator F_2 in the form:

$$F_2(T,x) = (2\beta)^{-1}\left\langle \frac{\delta u(0,0)}{\delta u(T,x)} \right\rangle . \tag{5}$$

The variational derivative $\Theta(t,y) = \delta u(t,y)/\delta u(T,x)$ for $\nu \to 0$ satisfies the continuity equation:

$$\Theta_t + u\Theta_y + u_y\Theta = 0, \tag{6}$$

and the condition $\Theta(T,y) = \delta(x-y)$. This Cauchu problem is solved by the characteristic method and we arrive to the expression for $F_2(T,x)$:

$$F_2(T,x) = (2\beta)^{-1}\langle\Theta(0,0)\rangle = (2\beta)^{-1}\left\langle \delta\left(x - y(T)\right)\left(\frac{\partial y(T,\zeta)}{\partial\zeta}\right)_{\zeta=0}\right\rangle, \tag{7}$$

(see [8]). Here $y(T,\zeta)$ is coordinate of the Lagrange particle started at the instant $t=0$ from the point ζ:

$$\dot{y} = u(t,y), \quad y(0,\zeta) = \zeta, \tag{8}$$

and $y(T) = y(T,0)$. If $u(t,y)$ is discontinuous, then the equation (8) requires a regularization. We use physically evident condition that the particle on a shock wave front moves with the velocity of this front. Its formal justification starting from a finite small viscosity can be found in [9].

The expression (7) tells us that the correlator F_2 in the limit being considered is determined by a most probable initial velocity fluctuation $u_0(y)$ which, evolving, carries out the particle from the point 0 to the point x in the time T. The probabilities of initial configurations are defined by the functional (3). The desired optimal fluctuation $u_0(y)$ minimizes $\mathcal{F}[u_0]$ with the condition $y(T) = x$. Let us show that it is the linear profile:

$$u_0(y) = u_0^* \equiv x/T - y/T,\ 0 < y < x, \quad u_0(y) = 0,\ y < 0,\ y > x. \tag{9}$$

First, it is evident that the function $u_0(y)$ must have a maximum at $y = 0$. It is also easy to understand the equality $u_0(y)$ to 0 for $y < 0$ and $y > x$.

Indeed, difference $u_0(y)$ from zero outside the interval $(0,x)$ does not affect the trajectory $y(t)$ but $\mathcal{F}[u_0]$ increases. The left edge of the distribution $u(t,x)$ for $\nu \to 0$ will be straight line with the slope $\sigma = 1/t$. Such the time dependence can be checked by direct substitution into Burgers equation; see also [10]. At $t = T$ the coordinate of the most rapid particle will be equal to x. Coordinates of the other particles from the interval $(0,x)$ will be precisely the same. Thus, for the class of initial distributions $u_0(y)$ described above the plot of the final function $u(T,y)$ has a form of triangle:

$$u(T,y) = y/T,\ 0 < y < x, \quad u_0(y) = 0,\ y < 0,\ y > x. \tag{10}$$

Now let us note that from the Burgers equation it follows:

$$d\mathcal{F}[u(t,y)]/dt = -2\nu \int dy\, u_y^2 \leq 0, \tag{11}$$

what means that:

$$\mathcal{F}[u_0(y)] \geq \mathcal{F}[u(T,y)]. \tag{12}$$

This inequality becomes strict one even for $\nu \to 0$ if shock waves were formed during the evolution. Consequently, the minimal admissible value of the functional $\mathcal{F}$ is equal to:

$$\mathcal{F}[u(T,y)] = x^3/3T^2. \tag{13}$$

The value of $\mathcal{F}$ on the function $u_0^*(y)$ coincides with (13). The exlusion of shocks in the time interval $(0,T)$ justified above makes the expression (9) the only possible.

Probability of the initial fluctuation (9) proportional to $\exp\left(-\beta\mathcal{F}[u_0(y)]\right)$ defines the exponential part of the asymptotics of the pair correlator F_2:

$$F_2(T,x) \sim \exp\left(-\frac{\beta x^3}{3T^2}\right). \tag{14}$$

It is worth noting that the multipiler $(\partial y(T,\zeta)/\partial\zeta)_{\zeta=0}$ of the δ-function in the formula (7) vanishes on the configuration (9), but it becomes different from zero under small variation of $u_0(y)$. In other words, this factor, along with unknown pre-exponential factor in the expression (14) as a whole, is determined by integration over variations δu of the initial velocity field with

respect to $u_0^*(y)$. The essential values of δu are small comparing with $u_0^*(y)$; the parameter of this smallness is $(\beta x^3/T^2)^{-1}$. However, the integration over δu cannot be reduced to the Gaussian one even in the limit $\beta x^3/T^2 \gg 1$. The point is that at $\nu \to 0$ the functional $\mathcal{F}[u]$ is not analytical on the class of initial velocity fields $u(y)$ obeying the constraint $y(T) = x$. The variation $\delta \mathcal{F}$ turns out to be of the first order in δu despite that inequality $\delta \mathcal{F} \geq 0$ holds. $\mathcal{F}[u]$ can be expanded in functional Taylor series in δu for $\delta u \ll \nu/x$ only. The corresponding analysis will be given in another paper and here we restrict ourselves to exponential asymptotics.

Noting that the initial linear profile (9) tranfers all the internal point of the interval $(0, x)$ by the time $t = T$ into the point x we conclude that up to pre-exponential factor:

$$F_{n+2} = \left\langle u(T,x) \prod_{j=1}^{n} u(0,y_j)\ u(0,0) \right\rangle \sim F_2(T,x) \sim \exp\left(-\frac{\beta x^3}{3T^2}\right). \qquad (15)$$

Here $0 < y_1 < y_2 \cdots < y_n$. It is obvious that the reducible part of this correlator is equal to zero. The same initial fluctuation $u_0^*(y)$ determines leading asymptotics of the correlator $\Phi_4 = \langle u(T,x) u(T, x+a_1) u(0,a) u(0,0) \rangle$ at $0 < a < x$ and $0 < a_1 \ll a$:

$$\Phi_4 \sim \exp\left(-\frac{\beta x^3}{3T^2}\right) \gg \Phi_{4,Gauss} \sim \exp\left(-\frac{2\beta x^3}{3T^2}\right). \qquad (16)$$

Here $\Phi_{4,Gauss}$ designates reducible part of Φ_4. To find Φ_4 as a function of the parameter a it is necessary to analyse the evolution of the perturbed linear profile. In this case the shock waves formation becomes inevitable what makes the problem more difficult. It is worth adding that a-dependence of the correlator Φ_4 may be related directly to the probability distribution function of velocity field gradients. In [11, 12] it was shown that the latter id defined by forming shocks. Proportionality of asymptotics of high order correlation functions to the asymptotics of pair correlator is characteristic for turbulent-like problems and in such the context was noted in [13].

Correlation functions of the field $u(t, x)$ may be represented in a form of functional integrals (see e.g., [14]). In the present paper such the integrals were computed in essence by the saddle-point method with the parameter $\beta x^3/T^2 \gg 1$ contained into the object to be averaged, but not inherent to the action. This approach goes back to the works of I.M.Lifshits (see in [15]). Later it was generalized to find high order correlators in equilibrium [16] and

strongly non-equilibrium problems [17, 18, 19, 12, 20, 21, 22]. Optimal fluctuation is called also as an instanton by analogy with quantum field theory. In the paper [23] long-time asymptotics of the current autocorrelation function has been computed for a disordered contact and the large observation time was used as a saddle-point parameter.

I am greately indebted to Michael Chertkov and Vladimir Lebedev for numerous discussions and advices. I am grateful to Gregory Falkovich and Thomas Spencer for backing investigations leading to this work. I would like to appreciate Michael Stepanov for useful remarks.

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